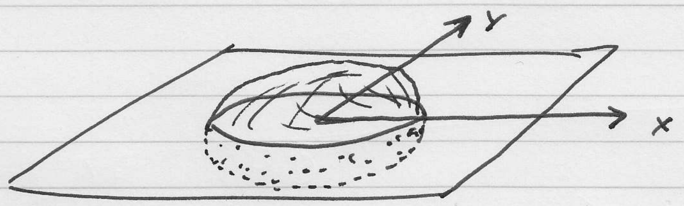
CONSIDER AN ELLIPSOID

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$$

both b^2.

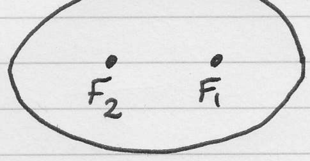
THIS ELLIPSOID IS SYMMETRIC ABOUT ROTATIONS ABOUT THE x-AXIS.

Consider the section of the ellipsoid within the xy plane



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \quad \text{ELLIPSE E1}$$

NEXT, CONSIDER THIS ELLIPSE IN POLAR COORDINATES.



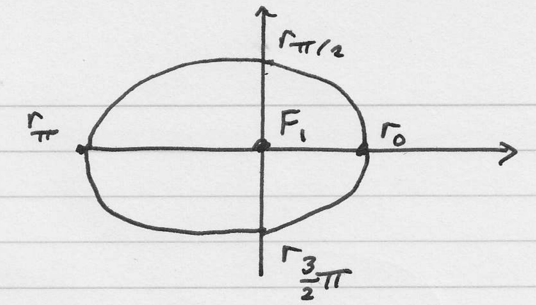
ORIGIN OF POLAR COORDINATE SYSTEM IS AT F_1.

$$r = r_0 \frac{1}{1 + A\cos\theta}$$

SCALE THE COORDINATE SYSTEM SUCH THAT $r_0 = 1$. THEN WE HAVE

$$r = \frac{1}{1 + A\cos\theta}$$

2



$F_1 = O$ = ORIGIN

FOUR POINTS ON THIS ELLIPSE, FOR WHICH $\theta \in \left\{ 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi \right\}$

HAVE $r = r_0$, $r = r_{\pi/2}$, $r = r_\pi$, $r = r_{\frac{3}{2}\pi}$,

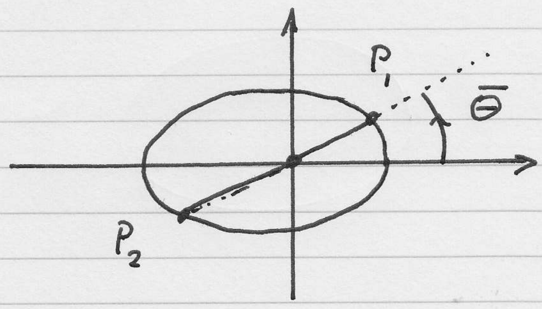
$$r_0 = \frac{1}{1+A}$$

$$r_{\pi/2} = 1$$

$$r_\pi = \frac{1}{1-A}$$

$$r_{3/2\pi} = 1$$

NEXT, SUPPOSE WE CONSIDER A POINT IN THE 1ST QUADRANT ($0 \leq \theta \leq \pi/2$) WITH $\theta = \bar{\theta}$

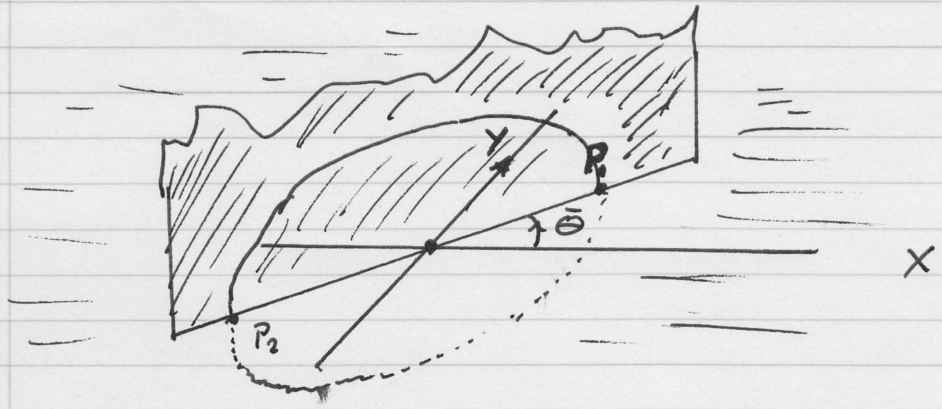


CHORD OF ELLIPSE, ANGLE $\bar{\theta}$ to X-AXIS, GIVEN BY LINE SEGMENT $\overline{P_1 P_2}$.

GOING BACK TO THE 3D-ELLIPSOID, WHAT IS SHAPE OF SECTION OF THE ELLIPSOID IN PLANE FORMED BY $\overleftrightarrow{P_1 P_2}$ AND Z-AXIS?

IT SHOULD BE AN ELLIPSE.
CALL THIS ELLIPSE E2.

QUESTION: IS F_1 ONE OF THE FOCI OF E2 ?



x and y axes here are not ORIGINAL x, y AXES - THEY ARE wrt ORIGIN IN POLAR COORDINATE SYSTEM.

Describe E2 USING A POLAR COORDINATE SYSTEM IN PLANE $\overleftrightarrow{P_1 P_2}$, z

polar coordinates r_α, θ_α.

E2 is described by

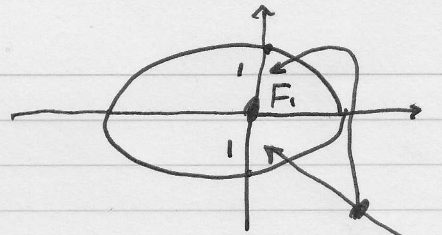
$$r_\alpha = r_{\alpha 0} \frac{1}{1 + A_{E2} \cos \theta_\alpha}$$

For $\theta_\alpha = 0$, $\quad r_\alpha = r_{\alpha 0} \frac{1}{1 + A_{E2}}$

$\theta_\alpha = \frac{\pi}{2}$, $\quad r_\alpha = r_{\alpha 0}$

$\theta_\alpha = \pi$, $\quad r_\alpha = r_{\alpha 0} \frac{1}{1 - A_{E2}}$

$\theta_\alpha = \frac{3\pi}{2}$, $\quad r_\alpha = r_{\alpha 0}$



ELLIPSE E1 IN XY PLANE

has points r, θ

$$= \left(1, \tfrac{\pi}{2}\right)$$

and

$$\left(1, \tfrac{3\pi}{2}\right)$$

THEREFORE, THESE DISTANCES WITH RESPECT TO THE POLAR X-Y AXES ARE EQUAL TO 1.

THEREFORE THE 3D ELLIPSOID ALSO CONTAINS THE POINTS $(0\ 0\ 1)$ AND $(0\ 0\ -1)$

WITH RESPECT TO THE ABOVE X, Y AXIS AND A Z-AXIS THROUGH F_1 AND PERPENDICULAR TO THE PAPER.

THEREFORE, FOR ELLIPSE E2,

$$r_{\alpha 0} = 1$$

AND THE FOUR POINTS ARE GIVEN BY

θ_α	r_α
0	$\dfrac{1}{1+A_{E2}}$
$\pi/2$	1
π	$\dfrac{1}{1-A_{E2}}$
$3\pi/2$	1

And FOR P_1, $\bar{r}$ IS GIVEN BY

$$\bar{r} = \frac{1}{1 + A\cos\bar{\theta}}$$

FOR P_2, $\bar{r}$ IS GIVEN BY

$$\bar{r} = \frac{1}{1 + A\cos(\bar{\theta} + \pi)} = \frac{1}{1 - A\cos\bar{\theta}}$$

so four points on E2 ARE GIVEN BY

$$\theta_2 = 0 \qquad r_2 = \frac{1}{1 + A\cos\bar{\theta}}$$

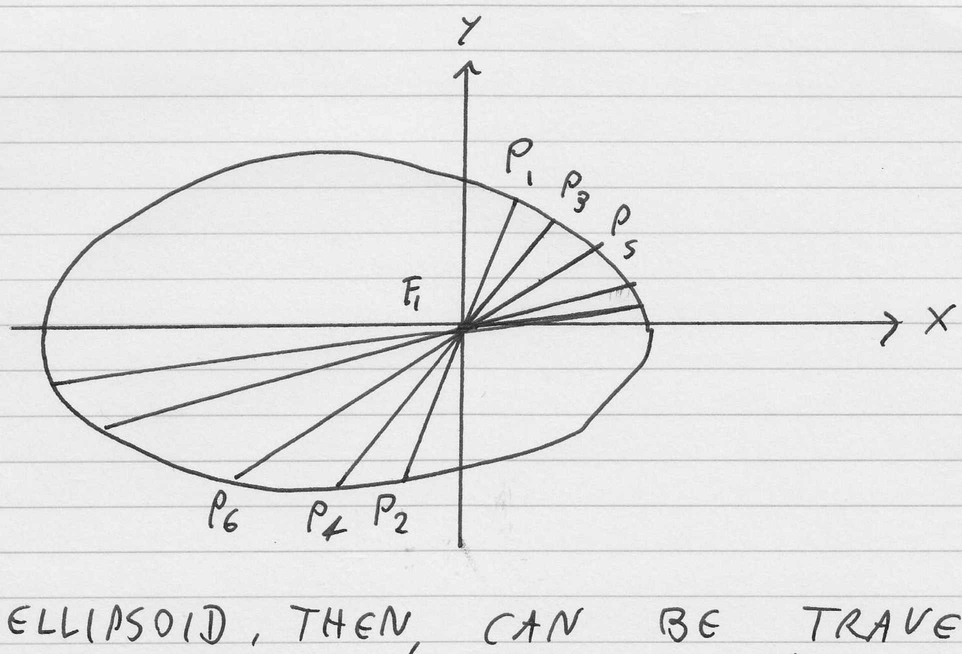
$$\theta_2 = \pi/2 \qquad r_2 = 1$$

$$\theta_2 = \pi \qquad r_2 = \frac{1}{1 - A\cos\bar{\theta}}$$

$$\theta_2 = 3\pi/2 \qquad r_2 = 1$$

THEREFORE, E2 IS INDEED AN ELLIPSE WITH F_1 AS ONE OF ITS FOCI, AND

$$A_{E2} = A\cos\bar{\theta}$$



THE ELLIPSOID, THEN, CAN BE TRAVERSED BY FILAMENTARY ELLIPSES IN PLANES $\perp$ XY, ABOVE AND BELOW $\overline{P_1P_2}$ $\overline{P_3P_4}$ $\overline{P_5P_6}$, etc...

THESE FILAMENTARY ELLIPSES EACH HAVE F_1 AS ONE OF THEIR FOCI.

AND WE CAN GRADUALLY ROTATE THIS PATTERN OF FILAMENTARY ELLIPSES TO GET A PATTERN ANALOGOUS TO THE ORBITSPHERE.

ROTATION ABOUT x - AXIS.

NET ANGULAR MOMENTUM IN $\vec{X}$ DIRECTION.